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Statistical Distribution of the Enhanced Backscatter Coefficient in Reverberation Chamber

Qian Xu, Lei Xing, Zhihao Tian, *Student Member*, *IEEE*, Yongjiu Zhao, Xinlei Chen, *Member*, *IEEE*, Long Shi, *Member*, *IEEE*, and Yi Huang, *Senior Member, IEEE*

***Abstract*—In an ideal RC, the analytical expressions for the statistical distributions of the enhanced backscatter coefficient (e_b) are derived. It has been found that the superposition of random variables with exponential distributions has a probability density function of Erlang distribution, and the distribution of the ratio of two random variables with Erlang distribution can be derived analytically, which is the distribution of the enhanced backscatter coefficient. The unbiased estimator of e_b is also given with and without approximations. Measurements have been performed to verify the results.**

***Index Terms*— statistical electromagnetics, enhanced backscatter coefficient, statistical distribution, reverberation chamber.**

I. Introduction

THE enhanced backscattering phenomenon is a universal effect in nearly all wave phenomenon [1-3]. In the reverberation chamber (RC) measurements, the enhanced backscatter coefficient e_b is an important parameter. Previous work has shown that the performance of the RC can be characterized by measuring e_b and checking how far it deviates from 2 (the theoretical value in an ideal RC) [3]. In the antenna efficiency measurement [4-6], e_b is required to be 2 if the one-antenna method [4] is used. However, since the measured e_b is a random variable, it cannot equal to 2 exactly. Thus, it is important to know the probability density function (PDF) of e_b, as a PDF means a complete understanding of a random variable. From the PDF, all the statistical quantities can be derived (e.g. expectation, standard deviation, moments, *etc*.), an unbiased estimator for e_b can also be obtained which can be used to estimate the mean value when the sample number is not large.

In this paper, we focus on the statistical distribution of e_b, the definition of e_b is briefly reviewed in Section II, in Section III, the derivation of the PDF under given preconditions is given, from the PDF, the unbiased estimator can also be obtained. In Section IV, measurements are performed to validate the results. Finally, discussion and conclusions are summarized.

II. Enhanced Backscatter Coefficient

A typical antenna measurement setup in an RC is illustrated in Fig. 1, where a computer is used to control the vector network analyzer (VNA) and the rotation of stirrers. In the mode-tuning mode, the stirrers rotate step by step. For each step of the stirrers, the computer records the measured *S*-parameters from the VNA.

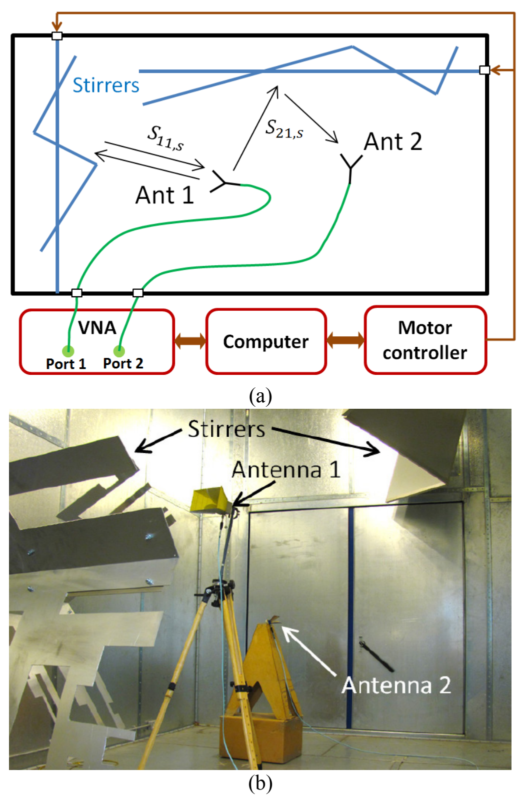

Fig. 1. Typical antenna measurement setup in an RC, (a) schematic plot, (b) measurement scenario.

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Q. Xu, L. Xing, Y. Zhao and X. Chen are with College of Electronic and Information Engineering, Nanjing University of Aeronautics and Astronautics, Nanjing 211106, China (e-mail: emxu@foxmail.com).

Z. Tian is with the College of Optoelectronic Science and Engineering, National University of Defense Technology, Changsha 410073, China.

L. Shi is with Science and Math Cluster, Singapore University of Technology and Design, 487372, Singapore.

Y. Huang is with the Department of Electrical Engineering and Electronics, The University of Liverpool, Liverpool L69 3GJ, UK. (e-mail: yi.huang@liv.ac.uk).

Suppose that Ant 1 is used as the transmitting antenna, and Ant 2 is far from Ant 1. The enhanced backscatter coefficient is defined as the ratio of the average received power of Ant 1 and Ant 2. The average can be taken over different stirrer positions or/and different frequencies. Mathematically, the enhanced backscatter coefficient at the position of Ant 1 is [4]

$$e_{b1} = \frac{\langle |S_{11,s}|^2 \rangle / \eta_1}{\langle |S_{21,s}|^2 \rangle / \eta_2} = \frac{\langle |S_{11} - \langle S_{11} \rangle|^2 \rangle / \eta_1}{\langle |S_{21} - \langle S_{21} \rangle|^2 \rangle / \eta_2} \quad (1)$$

where $S_{*,s}$ means the stirred part of the measured S_*, $\langle\cdot\rangle$ represents the average operation, η_1 and η_2 are the total efficiency of Ant 1 and Ant 2, respectively. Similarly, if Ant 2 is used as the transmitting antenna, at the position of Ant 2

$$e_{b2} = \frac{\langle |S_{22,s}|^2 \rangle / \eta_2}{\langle |S_{12,s}|^2 \rangle / \eta_1} = \frac{\langle |S_{22} - \langle S_{22} \rangle|^2 \rangle / \eta_2}{\langle |S_{12} - \langle S_{12} \rangle|^2 \rangle / \eta_1} \quad (2)$$

In [4], the enhanced back scatter coefficient is written as

$$e_b = \frac{\sqrt{\langle |S_{11,s}|^2 \rangle \langle |S_{22,s}|^2 \rangle}}{\langle |S_{21,s}|^2 \rangle} = \sqrt{e_{b1} e_{b2}} \quad (3)$$

which can be considered as the geometric mean value of e_{b1} and e_{b2}. Note that $e_{b1} = e_{b2} = e_b = 2$ [2, 4] in a well-stirred (ideal) RC, thus the enhanced backscatter coefficient can be used to characterize the performance of the RC [3].

III. Statistical Distribution

Existing work treated e_b as a constant. However, from (1), we can find that the enhanced backscatter coefficient is actually a random variable, as both the numerator and denominator are random variables. To investigate the analytical expression of the probability distribution of e_{b1}, some preconditions need to be made to simplify the analysis.

In (1), we assume that both Ant 1 and Ant 2 are high efficient antennas and $\eta_1 \approx \eta_2 \approx 100\%$, and the RC is well-stirred, from Hill's equation [7] we have

$$\langle |S_{21,s}|^2 \rangle = \frac{Q\lambda^3}{16\pi^2 V} \equiv T_0 \quad (4)$$

where Q is the quality factor of the RC, λ is the wavelength, and V is the volume of the RC. We denote the mean value of it as T_0 as the chamber transfer function, thus $|S_{21,s}|^2$ has the following exponential distribution [7]:

$$\mathrm{PDF}(X) = \frac{1}{T_0} \exp\left(-\frac{x}{T_0}\right), \quad X = |S_{21,s}|^2 \quad (5)$$

Since the RC is assumed to be well stirred and $\eta_1 \approx \eta_2 \approx 100\%$, we have $e_{b1} = 2$ in (1). Therefore, $|S_{11,s}|^2$ also has an exponential distribution but the mean value of which is $2T_0$. Without loss of generality, the PDF of $|S_{11,s}|^2$ has the form of

$$\mathrm{PDF}(X) = \frac{1}{\alpha T_0} \exp\left(-\frac{x}{\alpha T_0}\right), \quad X = |S_{11,s}|^2 \quad (6)$$

where $\alpha = 2$ for an ideal RC. Consider a sample number of N, Ref. [8, 9] have shown that the superposition of random variables with exponential distribution is the Erlang distribution, and the mean value of $|S_{21,s}|^2$ in (5) with N samples has a PDF of

$$\mathrm{PDF}(X) = \frac{N(Nx)^{N-1} \exp[-Nx/T_0]}{T_0^N (N-1)!}, X = \langle |S_{21,s}|^2 \rangle_N \quad (7)$$

The mean value of X in (7) is also T_0 and the standard deviation is $T_0/\sqrt{N}$. Similarly, $\langle |S_{11,s}|^2 \rangle_N$ has the same expression as (7) but T_0 is replaced by αT_0.

By investigating the Erlang distribution, we can find the PDF of the ratio of two random variables with Erlang distribution [7] as:

$$\mathrm{PDF}(X) = \frac{\alpha^N x^{N-1}}{B(N,N)(x+\alpha)^{2N}} = \frac{\Gamma(2N)\alpha^N x^{N-1}}{\Gamma(N)^2 (x+\alpha)^{2N}}$$

$$= \frac{(2N-1)!\, \alpha^N x^{N-1}}{[(N-1)!]^2 (x+\alpha)^{2N}}, X = \frac{\langle |S_{11,s}|^2 \rangle_N}{\langle |S_{21,s}|^2 \rangle_N} \quad (8)$$

where $B(v_1, v_2)$ is the Beta function [10]:

$$B(v_1, v_2) = \int_0^1 t^{v_1 - 1}(1-t)^{v_2 - 1} dt \quad (9)$$

and $\Gamma(x)$ is the Gamma function:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \quad (10)$$

(8) is actually the PDF of e_{b1} in (1). The mean value can be obtained as

$$\mathrm{E}(e_{b1N}) = \alpha N/(N-1) \quad (11)$$

The standard deviation can also be derived as

$$\mathrm{Std}(e_{b1N}) = \frac{\alpha\sqrt{N(2N-1)}}{(N-1)\sqrt{N-2}} \quad (12)$$

where e_{b1N} means e_{b1} is obtained from N sample number.

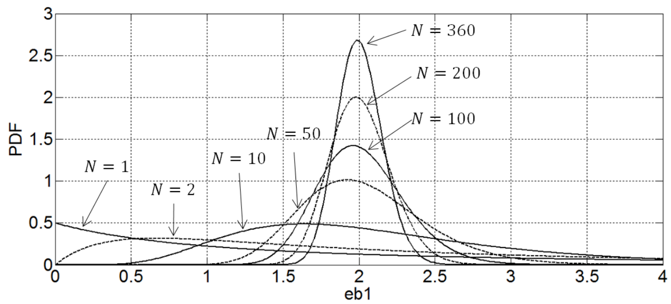

Fig. 2. PDF of e_{b1} with different sample number N, $\alpha = 2$.

From (11), we can solve for the unbiased estimator of e_{b1} (α):

$$\alpha = \bar{e}_{b1}\frac{N-1}{N} \tag{14}$$

where $\bar{e}_{b1}$ is the ensemble average, of which each sample is obtained by using N stirrer positions in the mechanical stir. The PDF in (8) with different sample number is illustrated in Fig. 2.

Note that frequency stir can also be used in (1) to increase the sample number, but there is a subtle difference between the frequency stir and smoothing over frequencies after calculation using (1). The frequency stir is applied to obtain the stirred part of S-parameters $S_{*,s}$, while smoothing over frequencies for (1) means the average value for many random variables with the same distribution in (8). Suppose that the random variable X is defined as the smoothed e_{b1} with M samples

$$X = \langle\frac{\langle|S_{11,s}|^2\rangle_N}{\langle|S_{21,s}|^2\rangle_N}\rangle_M \tag{15}$$

The PDF of X in (15) has no simple analytical expression (the integral cannot be expressed in terms of elementary functions), but we can find the mean and the standard deviation as

$$\mathrm{E}\left(\langle\frac{\langle|S_{11,s}|^2\rangle_N}{\langle|S_{21,s}|^2\rangle_N}\rangle_M\right) = \alpha\frac{N}{N-1} \tag{16}$$

$$\mathrm{Std}\left(\langle\frac{\langle|S_{11,s}|^2\rangle_N}{\langle|S_{21,s}|^2\rangle_N}\rangle_M\right) = \frac{\alpha\sqrt{N(2N-1)}}{(N-1)\sqrt{M(N-2)}} \tag{17}$$

It is well-known that for a large value of M, X in (15) has a normal distribution because of the central limit theorem (CLT) [11].

For e_{b2} in (2), the expressions are the same. We can move a step closer to the PDF of e_b in (3) by finding the PDF of the numerator in (3). We found that the PDF of the square root of the product of two independent random variables with the same Erlang distribution in (7) has an analytical expression:

$$\mathrm{PDF}(X) = \frac{4N(Nx)^{2N-1}K_0(\frac{2Nx}{\alpha T_0})}{\Gamma(N)^2(\alpha T_0)^{2N}},$$

$$X = \sqrt{\langle|S_{11,s}|^2\rangle_N\langle|S_{22,s}|^2\rangle_N} \tag{18}$$

where $K_0(v)$ is the zero-order modified Bessel functions of the second kind

$$K_0(v) = \int_0^{\infty}\frac{cos(vt)}{\sqrt{t^2+1}}dt \tag{19}$$

Then we can derive the mean value and the standard deviation of PDF in (18) as

$$\mathrm{E}(X) = \frac{\alpha T_0\Gamma(N+1/2)^2}{N\Gamma(N)^2} \tag{20}$$

$$\mathrm{Std}(X) = \alpha T_0\sqrt{1-\frac{\Gamma(N+1/2)^4}{N^2\Gamma(N)^4}} \tag{21}$$

Finally, from (7) and (18), we can further derive the PDF of e_b in (3) as

$$\mathrm{PDF}(X) = \frac{\alpha^N}{x^{N+1}\Gamma(N)^3}\left[h_1(x)\Gamma\left(\frac{3N}{2}\right)^2 - \frac{\alpha}{x}h_2(x)\Gamma\left(\frac{1}{2}+\frac{3N}{2}\right)^2\right],$$

$$X = \frac{\sqrt{\langle|S_{11,s}|^2\rangle_N\langle|S_{22,s}|^2\rangle_N}}{\langle|S_{21,s}|^2\rangle_N} = e_{bN}$$

where

$$h_1(x) = F_{2,1}\left(\left[\frac{3N}{2},\frac{3N}{2}\right],\frac{1}{2},\frac{\alpha^2}{4x^2}\right) = \sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{3N}{2}+n\right)^2\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3N}{2}\right)^2\Gamma\left(\frac{1}{2}+n\right)n!}\left(\frac{\alpha^2}{4x^2}\right)^n$$

$$h_2(x) = F_{2,1}\left(\left[\frac{3N+1}{2},\frac{3N+1}{2}\right],\frac{3}{2},\frac{\alpha^2}{4x^2}\right) = \sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{3N+1}{2}+n\right)^2\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3N+1}{2}\right)^2\Gamma\left(\frac{3}{2}+n\right)n!}\left(\frac{\alpha^2}{4x^2}\right)^n \tag{22}$$

where $F_{2,1}$ represents the generalized hypergeometric function [10]. The mean value and the standard deviation of e_{bN} are

$$\mathrm{E}(e_{bN}) = \frac{\alpha\Gamma\left(N+\frac{1}{2}\right)^2}{(N-1)\Gamma(N)^2} \tag{23}$$

$$\mathrm{Std}(e_{bN}) = \frac{\alpha}{N-1}\sqrt{N^2\frac{N-1}{N-2} - \frac{\Gamma(N+1/2)^4}{\Gamma(N)^4}} \tag{24}$$

The PDF in (22) with different sample number N is illustrated in Fig. 3, and the cumulative distribution functions (CDFs) are given in Fig. 4. The CDFs are calculated numerically from the PDF, as simple expressions are not available.

When N is large [10]

$$\lim_{N\to\infty}\frac{\Gamma(N+t)}{\Gamma(N)N^t} = 1 \tag{25}$$

we can approximate (23) and (24) by using:

$$\mathrm{E}(e_{bN}) \approx \frac{\alpha N}{N-1} \tag{26}$$

$$\mathrm{Std}(e_{bN}) \approx \frac{\alpha N}{(N-1)\sqrt{N-2}} \tag{27}$$

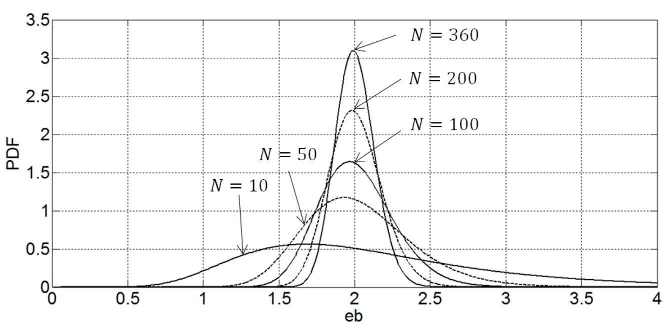

Fig. 3. PDF of e_b with different sample number N, $\alpha = 2$.

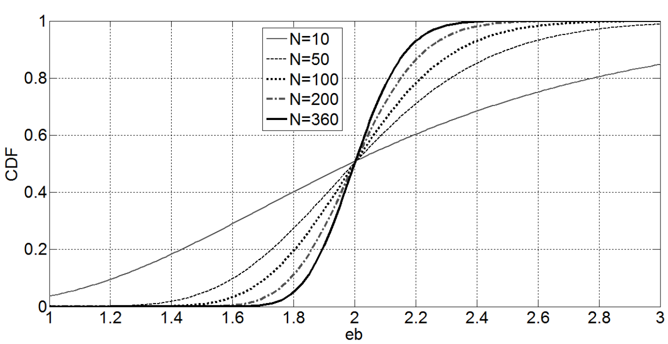

Fig. 4. CDF of e_b with different sample numbers N, $\alpha = 2$.

Figure 5 shows the mean and the standard deviations in (23), (24) and (26), (27). As can be seen, when N is large, the gap between the approximated value and exact value becomes negligible. Also, from (23) and (26), we can obtain the unbiased estimator for e_b as

$$\alpha = \bar{e}_{bN}\frac{(N-1)\Gamma(N)^2}{\Gamma\left(N+\frac{1}{2}\right)^2} \tag{28}$$

or

$$\alpha \approx \bar{e}_{bN}\frac{N-1}{N}, \text{when } N \text{ is large} \tag{29}$$

where $\bar{e}_{bN}$ is the ensemble average, of which each sample is obtained by using N stirrer positions in the mechanical stir.

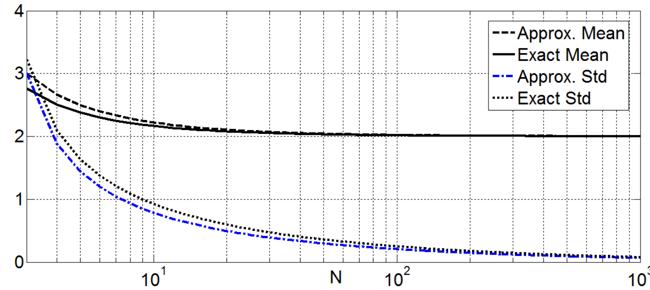

Fig. 5. The exact and approximate means and standard deviations of e_{bN}, with $\alpha = 2$.

Probabilities of e_{bN} in different ranges are calculated and tabulated in Table I. As can be seen, when N is large, the probability in the given range converges to around 68%, 95% and 99% for ±1, ±2 and ±3 standard deviations, respectively.

TABLE I
PROBABILITIES OF e_{bN}

N	Range	Probability
$N = 10$	$\mathrm{E}(e_{bN}) \pm \mathrm{Std}(e_{bN})$	75.3%
	$\mathrm{E}(e_{bN}) \pm 2\mathrm{Std}(e_{bN})$	95.7%
	$\mathrm{E}(e_{bN}) \pm 3\mathrm{Std}(e_{bN})$	98.9%
$N = 50$	$\mathrm{E}(e_{bN}) \pm \mathrm{Std}(e_{bN})$	69.5%
	$\mathrm{E}(e_{bN}) \pm 2\mathrm{Std}(e_{bN})$	95.7%
	$\mathrm{E}(e_{bN}) \pm 3\mathrm{Std}(e_{bN})$	99.3%
$N = 100$	$\mathrm{E}(e_{bN}) \pm \mathrm{Std}(e_{bN})$	68.9%
	$\mathrm{E}(e_{bN}) \pm 2\mathrm{Std}(e_{bN})$	95.6%
	$\mathrm{E}(e_{bN}) \pm 3\mathrm{Std}(e_{bN})$	99.5%
$N = 360$	$\mathrm{E}(e_{bN}) \pm \mathrm{Std}(e_{bN})$	68.4%
	$\mathrm{E}(e_{bN}) \pm 2\mathrm{Std}(e_{bN})$	95.5%
	$\mathrm{E}(e_{bN}) \pm 3\mathrm{Std}(e_{bN})$	99.7%

IV. MEASUREMENT

To verify the theoretical distribution in (22), we have to note that in a practical RC, α may not be exactly 2. Thus, we can normalize e_{bN} to α in (28) to obtain a normalized PDF which is independent of α. By applying the normalization, the measured CDF and the theoretical CDF can be compared. From (22), the normalized PDF can be obtained as

$$\mathrm{PDF}(X) = \frac{1}{x^{N+1}\Gamma(N)^3}\left[h_1{}'(x)\Gamma\left(\frac{3N}{2}\right)^2 - \frac{1}{x}h_2{}'(x)\Gamma\left(\frac{1}{2}+\frac{3N}{2}\right)^2\right],$$

$$X = \frac{\sqrt{\langle|S_{11,s}|^2\rangle_N\langle|S_{22,s}|^2\rangle_N}}{\alpha\langle|S_{21,s}|^2\rangle_N} = \frac{e_{bN}}{\alpha}$$

where

$$h_1'(x) = F_{2,1}\left(\left[\frac{3N}{2},\frac{3N}{2}\right],\frac{1}{2},\frac{1}{4x^2}\right)$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{3N}{2}+n\right)^2 \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3N}{2}\right)^2 \Gamma\left(\frac{1}{2}+n\right) n!}\left(\frac{1}{4x^2}\right)^n$$

$$h_2'(x) = F_{2,1}\left(\left[\frac{3N+1}{2},\frac{3N+1}{2}\right],\frac{3}{2},\frac{1}{4x^2}\right)$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{3N+1}{2}+n\right)^2 \Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3N+1}{2}\right)^2 \Gamma\left(\frac{3}{2}+n\right) n!}\left(\frac{1}{4x^2}\right)^n \quad (30)$$

which is the same as (22) when $\alpha = 1$.

Measurement data in [12] (RC at the University of Liverpool) are used to verify the results, the measurement setup is given in Fig. 1(b). Two horn antennas were used as Ant 1 (Rohde & Schwarz® HF 906) and Ant 2 (SATIMO® SH2000). 10, 50, 100 and 360 stirrer positions were used. At each stirrer position, 10001 frequency points were collected in the frequency range of 2.8 GHz to 4.2 GHz. The measured e_b samples are calculated using (3) and illustrated in Fig. 6. All the measured e_b samples in 2.8 GHz ~ 4.2 GHz are used to plot the CDF, the normalized theoretical and measured CDF are shown in Fig. 7, as expected, a very good agreement is obtained. When $N = 50$, 100 and 360 a small deviation is observed for CDF<0.001 because of finite samples. When $N = 10$, because the sample number is small, a small deviation is observed for CDF<0.1. We have also used the Kolmogorov–Smirnov (KS) test to verify the distribution, not surprisingly, for all N samples used in this paper, the results failed to reject the hypothesis at a significance level of 0.05.

From the results in this paper, we can have a complete statement of the measured e_b. Take $N = 360$ as an example, the measured average value of $\bar{e}_{bN}$ in the whole frequency range is 2.07, and the estimated α can be obtained by using (28), which is

$$\alpha = \bar{e}_{bN}\frac{359\Gamma(360)^2}{\Gamma(360.5)^2} \approx 0.9979\bar{e}_{bN} \approx 2.066 \quad (31)$$

The standard deviation can be calculated by using (24) and considering the independent frequency samples [7]

$$M \approx \frac{BW}{\Delta f} \approx \frac{BW}{f/Q} = \frac{BW}{f/(\omega\tau)} = 2\pi\tau BW \quad (32)$$

where $BW = 1.4$ GHz is the averaging bandwidth, τ is the chamber decay constant which can be extracted by applying the inverse Fourier transform to the measured S-parameters [4]. In 2.8 GHz ~ 4.2 GHz, the minimum τ is 1186 ns. From (24), the standard deviation of each sample is about 0.134, thus the standard deviation of the average value over the whole frequency band is

$$\text{std} = \frac{0.134}{\sqrt{M}} \approx \frac{0.134}{\sqrt{2 \times 3.14 \times 1186 \times 1.4}} \approx 0.0013 \quad (33)$$

Finally, a standard uncertainty statement can be given: the measured average e_b in the frequency range of 2.8 GHz ~ 4.2 GHz is believed to lie in the interval of 2.066 ± 0.0039 with a level of confidence of approximately 99.7%.

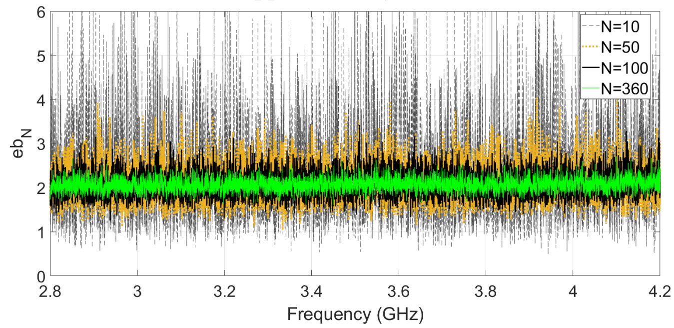

Fig. 6. Measured e_{bN} samples when $N = 10$, 50, 100 and 360.

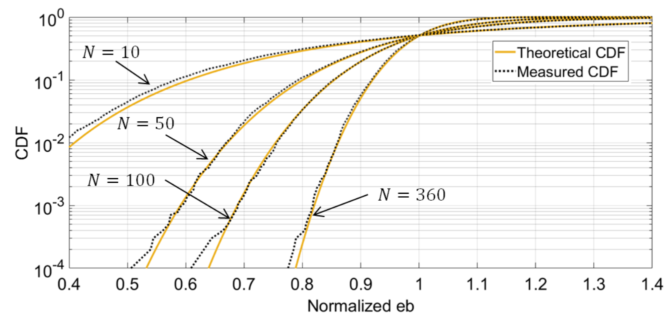

Fig. 7. A comparison of normalized theoretical and measured CDF of e_{bN}, $N = 10$, 50, 100 and 360.

V. Discussion and Conclusions

This paper investigated the statistical distribution of enhanced backscatter coefficient (e_b) and derived the analytical expressions for e_{b1}, e_{b2} and e_b. We proposed the unbiased estimator for e_b together with approximated expressions. Existing measurement data have been used to verify the analytical distribution of e_b.

Note that the derived expressions are rigorous for an ideal RC, the central limit theorem (CLT) is not used in the derivation. Actually, if the CLT is used, no simple expressions can be obtained as the integral cannot be evaluated in a closed form. We have also tried to use normal distributions to approximate $\langle|S_{11,s}|^2\rangle_N$, $\langle|S_{22,s}|^2\rangle_N$ and $\langle|S_{21,s}|^2\rangle_N$, but no simple analytical expression can be obtained for e_{bN}.

By using the results in this paper, one can evaluate the performance of RC using e_b confidently: e.g. identify the statistical variation interval of the measured e_b, evaluate the uncertainty and estimate it unbiasedly.

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